L4) Mathematics Education in the Lower School.
Dr Robert Rose

This power point concentrates on Steiner's philosophy of mathematics education. It has to be kept in mind that this power point does not teach the mathematics involved, nor the complete mathematics curriculum, but the underlying ***philosophy*** of how to approach mathematics education in Steiner Lower Schools. What this primarily consists in is the correlation between the change of sub-phase of child development with the content of the curriculum and the human learning faculty appealed to in the learning process. Moreover, where practical examples are given here, they are intended to illustrate certain points of principle rather than be parts of an exhaustive curriculum.

In addition, there is much that a teacher will need to learn in terms of mathematics itself as well as the more complete curriculum content and structure. The latter of these is self-created in relationship to the underlying philosophy as well as to the practical indications given in the primary and secondary texts. For those who need to revise their mathematics knowledge, a study of books covering the age range 7 to 14 would help (see below). This is the approximate equivalent to key stages 2 and 3 in mainstream education. Such texts might be helpful to Steiner Waldorf Teachers as a starting points, but will need substantial transformation in adapting them to Steiner's educational philosophy.

Mathematics education in Steiner / Waldorf schools is cultivated across all ages from class 1 to 12, but the focus here is on the lower school. As before, the content and the method of teaching is adapted to the different phases, and sub-phases, of child development. In terms of content, there are three general areas of mathematics covered, but with the usual subdivisions: **Arithmetic**, **Algebra** and **Geometry**. See **summary table** later for more details. Particularly relevant reading for mathematics education in Steiner / Waldorf Schools:

Avison, K & Rawson, M (eds)(2014): *The Educational Tasks and Content of the Steiner Waldorf Curriculum*, Floris Books, chapter 9.

Rawson, M, Burnett, J and Mepham, T (eds) (1999) Steiner Waldorf Education in the UK – Aims, Methods and Curriculum. (Mathematics Components)

Jarman, R (1998): Teaching Mathematics in Rudolf Steiner Schools for Classes I – VIII, Hawthorne Press.

Steiner, (1919): Discussions with Teachers, Anthroposophic Press, chapters 13 & 14.

For those who wish to revise their mathematics knowledge and / or make a comparison with mainstream mathematics education you could study:

Gordon, K (2009): Collins Revision: Key Stage 3 Maths, Levels 5-8, Harper Collins, London. (For ages 11-14 years) and its companion book for Key Stage 2 (ages 7 – 11 years).

Mathematics and the Teacher's Frame of Mind

For Steiner, the teaching of mathematics is not just concerned with the transmission of mathematical concepts to children. It is also about the frame of mind of the teacher and how this can make a connection to the inner life of the pupils:

"A boring math teacher will achieve very little if anything at all, whereas teachers who are inspired by this subject will succeed in making it stimulating and exhilarating. After all, it is by the grace of mathematics that, fundamentally, we can experience the **harmonies of ideal space**. If teachers can become **enthusiastic** about the Pythagorean theorem or the inner harmonies between planes and solids, they bring something into lessons that has immense importance for children, even in terms of soul development. In this way, teachers counteract the elements of confusion that life presents". Steiner, R (1922): Soul Economy, Anthroposophic press, p. 207. (My bold)

One can see in this that Steiner made the case that the teacher's enthusiasm not only makes the learning of mathematical content more possible but also that the confusion that may exist in the life of the child can be overcome through the pupil's inner participation in the harmonies of ideal space that mathematics can exemplify.

Mathematics and the Child's Soul Forces

Similarly, Steiner considered that the actual **act of observing** of mathematical forms can positively influence the thought and feeling of the child and can help them become a part of the world in a healthy way:

"You see, language could not exist without the constantly intermingling elements of thought and feeling. Again I have made an extreme statement, but if you examine various languages, you will discover how feeling and thinking are interwoven everywhere. This in itself, as well as many other factors, could easily introduce chaos into our lives were it not for the **inner firmness** that mathematics can give us. Those who can look more deeply into life know that many people have been saved from neurasthenia, hysteria, and worse afflictions simply by learning how to **observe** triangles, quadrilaterals, tetrahedra, and other geometrical realities in the right way... and so we can speak of the two poles in human development: the **rhythmic and artistic pole** and the **mathematical and conceptual one**. If, as indicated, young souls are worked on from within outward, students will gradually grow into the world in the right way." Steiner, R (1922): Soul Economy, Anthroposophic press, p. 207/8. (My bold)

Mathematics and the Practical Life

Following on from this, Steiner was keen that the teaching of mathematics is closely connection to the practical life:

"Your method must never be simply to occupy the children with examples you figure out for them, but you should give them practical examples from real life; you must let everything lead into practical life. In this way you can always demonstrate how what you begin with is fructified by what follows and vice versa." Steiner, R (1919): Discussions with Teachers, Anthroposophic Press, p. 156.

The aim in this particular context is that the children begin to see that learning is deeply related to real things in the World. In this case, mathematics, with its prevalence towards abstraction, is shown to be important for everyday life. Examples of this is in architechture, buying and selling, borrowing, interest, etc.

Mathematics and Moral Ideals

For Steiner, mathematics also has an effect on the moral life rather than it being confined merely to its own domain:

"Arithmetic and moral principles are two things that, in terms of logic, seem very removed from each other. It is not common to connect arithmetic with moral principles, because the logical connection is not obvious. It is obvious, however, to those who do not view the matter in terms of logic but in a living way that **children who are introduced to arithmetic correctly** will have a very different feeling of **moral responsibility** than those who were not. Now, what I am about to say may seem like an extreme paradox to you, but since I am speaking of realities and not of the illusions of our age, I will not fear the seemingly paradoxical, for these days truth often seems paradoxical. If people had known how to permeate the soul with mathematics in the right way during these past years, we would not now have Bolshevism in Eastern Europe. One perceives the forces that connect the faculty used in arithmetic with the springs of morality in humankind." Steiner, R (1922): The Spiritual Ground of Education, Anthroposophic Press, p. 76. (My bold)

Mathematics and the Phases of Child Development

As can be seen from the following quote, Steiner conceived mathematics education as being conditioned by different pedagogical methods in respect of the phases of child development:

"Throughout these stages, teachers should present mathematical elements in their manifold forms, in a **way appropriate to the student's age**. Mathematics, as taught in arithmetic and geometry, is likely to cause particular difficulties for teachers. Before the ninth year, this is introduced in simpler forms and subsequently expanded, since children can take in a great deal if we know how to go about it. It is a fact that all mathematical material taught throughout the school years must be presented in a thoroughly **artistic and imaginative way**. Using all kinds of means teachers must contrive to introduce arithmetic and geometry artistically, and here, too, between the ninth and tenth years teachers must go to a **descriptive method**. Students must be taught how to **observe** angles, triangles, quadrilaterals, and so on through a **descriptive method**. **Proofs** should not be introduced before the twelfth year." Steiner, R (1922): Soul Economy, Anthroposophic press, p. 206/7. (My bold)

Using our previous knowledge from the modules 3 and 5, we can represent this in the following way:

Subject	Sub Phase 1 (7th to 10th yrs) Classes 1-3	Sub Phase 2 (10th to 12th yrs) Classes 3-6	Sub Phase 3 (12th to 14th yrs) Classes 6-8
Mathematics	**Imaginative Anthropomorphisms** in the context of mathematics.	**Descriptive Method:** in the **observation, drawing, painting and modelling** of mathematical forms.	**Proof as Method**: using a visual demonstration approach.
	All within the General Imaginative / Artistic Method		

This can be connected to the lower school mathematics curriculum as outlined by Steiner in "Discussions with Teachers", First, Second and Third Lectures on the Curriculum.

Sub-Phase 1: From the Wholes to the Parts – and back again (Classes 1-3).

The first teaching & learning method in the lower school is the **artistic / imaginative approach**, often utilised in conjunction with the idea of **anthropomorphisms** as first introduced in module 3. In this early phase, Steiner / Waldorf teachers frequently introduce mathematical topics in “human form” such that numbers and the arithmetic processes are introduced as “people” with all the usual human characteristics but with an emphasis on those related to mathematical qualities . This enables the children to make a human connection in accordance with their stage of development. Obviously, this should not be at the cost of the veracity of the mathematical content, but rather should be an imaginative form which enables the assimilation of mathematics in accordance with the child’s being. This activity can be done in story form and each teacher is encouraged to create their own human stories specific to mathematics. At the same time, the content of the curriculum is taught in a number of holistic ways which we will see below. In the following, we will concentrate on this last aspect as the story form is special to each teacher.

Counting from the Whole

Once again, Steiner's approach to this subject matter is holistic in a number of different ways. He connects this onto the processes of **Synthesis** and **Analysis** and links these with the question of Human **Freedom**:

"In contrast, the soul aspect of all of our deeds is based upon analysis, which enables us to develop freedom in the life of pure thinking. If I am to find the sum of two and five and three, I have no freedom. There is a rule that dictates how much two and five and three are. On the other hand, if I have ten, then I can represent this number ten as nine plus one, or five plus five or three plus five plus two, and so forth. When analyzing, I carry out a completely free inner activity. When synthesizing, I am required by the external world to unfold the life in my soul in a particular way." Steiner, R (1920/2001): *The Renewal of Education*, Anthroposophic Press p. 168.

"We need to become aware of what actually wants to develop out of the child's individuality. **First** we need to know what can be drawn out of the child. At the outset children have a desire to be satisfied analytically; **then** they want to bring that analysis together through synthesis. We must take these things into account by looking at human nature." Steiner, R (1920/2001): *The Renewal of Education*, Anthroposophic Press p. 174.

Once again we see here that Steiner emphasises the dual aspect of analysis and synthesis. **It is only the combination of the both of these that characterises the wholeness of the mathematical activity, content and processes.**

We will return to the connection between analysis and synthesis shortly, but for now it needs to be mentioned that in the sequence of mathematics teaching in Steiner / Waldorf Schools, **counting** comes before arithmetic. Even so, Steiner's approach is still holistic in the sense that the process proceeds from the **whole human being or from some other whole being**:

"I should like to emphasize that this method of counting, **real counting**, should be presented **before** the children learn to do sums. They ought to be familiar with this kind of counting before you go on to arithmetic. Steiner, R (1924): The Kingdom of Childhood, Anthroposophic Press, p. 78. (My bold)

"Now all kinds of theories are thought out for the teaching of numbers and counting, and it is customary to act upon such theories. But even when external results can be obtained, the **whole being** of the child is not touched with this kind of counting or with similar things that have no connection with **real life**". Steiner, R (1924): The Kingdom of Childhood, Anthroposophic Press, p. 72. (My bold)

Steiner goes on to discuss how numbers can be derived from the whole human being in terms of a child's experience of their own body.

He describes how the teacher could show to the children how they, or their bodies, are a **One**. Then drawing the children's attention to their hands, a **Two** can be obtained, the fingers are a **Five** and a **Ten**, thereafter to legs, feet and toes, giving **Twenty.** Then he describes how counting can be learned from rhythmic stamping which necessarily includes the movement of the whole body (see chapter 5 of referenced text). He concludes:

In this way we bring **rhythm** into the series of numbers, and thereby too we foster the child's faculty of comprehending the **thing as a whole**. This is the natural way of teaching the children numbers, out of the reality of what numbers are. For people generally think that numbers were thought out by adding one to the other… The truth is that we count subconsciously on our fingers. In reality we count from one to ten on our ten fingers, then eleven (adding the toes), twelve, thirteen, fourteen (counting on the toes). You cannot see what you are doing, but you go up to twenty. And what you do in this manner with your fingers and toes only throws its reflection into the head. The head only looks on at all that occurs. The head is really only an apparatus for reflecting what the body does. The body thinks, the body counts. The head is only a spectator… **So you come to understand the nature of counting by actually *looking* at external objects.** You should develop the child's thinking by means of external things that can be seen, and keep as far away as possible from abstract ideas. The children can then gradually learn the numbers up to a certain point, first, let us say, up to twenty, then up to a hundred and so on. If you proceed on these lines you will be teaching them to count in a living way." Steiner, R (1924): The Kingdom of Childhood, Anthroposophic Press, pp. 75-78. (My bold and italics)

The Four Arithmetic Processes

From this one can lead counting onto the four arithmetic processes of **adding**, **multiplication**, **subtraction** and **division**:

"Instead of offering, say, three apples, then four more, and finally another two, and asking the child to add them all together, we begin by offering a **whole pile** of apples, or whatever is convenient. This would begin the whole operation. Then one calls on two more children and says to the first, "Here you have a pile of apples. Give some to the other two children and keep some for yourself, but each of you must end up with the same number of apples." In this way you help children comprehend the idea of sharing by three. We begin with the total amount and lead to the principle of division... According to our picture of the human being, and in order to attune ourselves to the children's nature, we do not **begin** by adding but by **dividing** and **subtracting**. **Then**, **retracing our steps and reversing the first two processes**, we are led to **multiplication** and **addition**. Moving **from the whole to the part**, we follow the original experience of number, which was one of **analyzing**, or division, and not the contemporary method of **synthesizing**, or putting things together by adding". Steiner, R (1921/22): Soul Economy, Anthroposophic Press, p. 149/50. (My bold).

Again, Steiner conceived these also as holistic activities in which teaching begins with the whole, proceeds to the parts and then back to the whole:

This can be summarised as:

Start from the Whole and proceed to the Parts.

Return from the Parts to the Whole.

Analysis

Synthesis

Division and Subtraction

Multiplication and Addition

Try the following as a practical exercise, maybe extend it by using your own creations. Try also the opposite approach (parts of wholes) and attempt develop a sense of how, or if, they generate a feeling of free activity:

	Answer
10 = 5 + ?	
9 = 10 - ?	
2 = 10 / ?	
20 = 2 x ?	

We can look at this in a more general way; do try these out giving as many examples as possible:

	Possible Answers
10 = ? + ?	
9 = ? - ?	
2 = ? / ?	
20 = ? x ?	

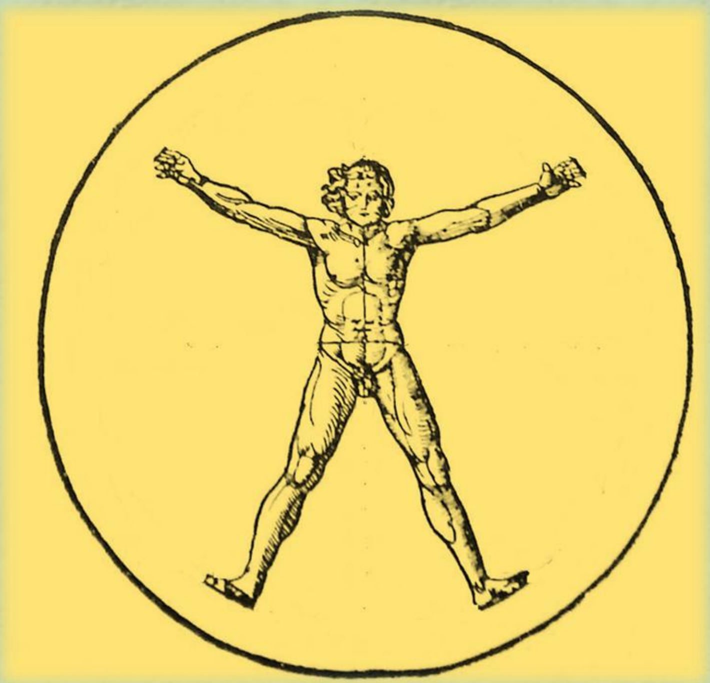
Steiner expands on the teaching of the multiplication process towards the multiplication tables:

"Let us look at ways of putting these ideas into practice. We can introduce children to the four rules of arithmetic as described in the previous lecture. We can give them some understanding of number relationships according to whether we subtract, divide, add, or multiply, as shown yesterday. But there is always an opportunity of letting students memorize **multiplication tables**, as long as these are related in the right way to the four rules. This also helps them deal with more complicated number relationships that will be introduced later." Steiner, R (1921/22): Soul Economy, Anthroposophic Press, p. 172. (My bold)

As is well known, the multiplication table can be represented as in the following image; some division processes can also be derived from this as can some powers and roots; the lilac diagonal is what you might call the powers and roots diagonal for integers in this range:

	The 10 Times Multiplication Table									
	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Try this short exercise using a self-created rhythmic movement method; using the text below as a starting point, see if you can create a rhythmic poem or song to which a class can do movements whilst counting. After this, devise a short lesson element to help the children do the maths involved in the text or poem (you will need to think of some related arithmetic activities and make some assumptions):



A person is **One**;
Their hands are **Two**;
The holes in their nose and mouth are **Three**,
Hands and Feet are **Four;**
Their fingers are **Five** and **Five** which are **Ten**;
As are their toes on their feet;
For fingers and toes there are **Twenty**;
How many children in the class?
How many children in the school?
How many then are there of fingers and toes in class and school?

Sub Phase 2 (Classes 3-6)

Dividing the Whole: Fractions

In class 4, fractions can be introduced, this again begins with a holistic approach:

"Of course there are an uncountable number of details to mention. Consider for a moment how appropriate it would be to include my characterization of arithmetic—to place analytical methods alongside synthetic methods, and to work with the sum and products and not simply from adding and factoring—along with what is normally done. You can see how appropriate it would be to treat **fractions** and everything connected with them from this perspective. When we move from working with whole numbers to working with fractions, we move in a quite natural way into the analytical. Moving from whole numbers to fractions means just that: analyzing. It is therefore appropriate to bring in another element when working with fractions than we use when working with whole numbers." Steiner, R (1920/2001): *The Renewal of Education*, Anthroposophic Press p. 231. (My bold)

Steiner conceived fractions as a product of a whole number divided by another which did not result in another whole number. Clearly, 2/2 = 1, so, obviously, this is not a fraction. But 1/2 obviously is.

Fractions, seen from a holistic perspective, are parts of wholes where the outcome is not a whole number. In this the analytic process plays a somewhat different role from previously. In the example discussed before, when an analysis is carried out a whole number is divided by another whole number and the outcome is a smaller whole number. With fractions this is clearly not the case, you are left with something a little unusual: something less than a whole number. We obviously call this a fraction. In the synthetic part of the process, the fractions can add up to a whole number, as is well known.

Visual demonstrations can be an aid, such as dividing one apple amongst a number of students, or a cake amongst the whole class. In the first case, say one apple is divided amongst four students, this, as is known, is what we call a quarter. Adding / synthesising the four quarters clearly gives one whole apple (mathematically speaking). Many practical exercises can be devised in order to give the children the experience of the full breadth of fractions and the application of the four arithmetic processes.

The following is an artistic image concerned with fractions:

Percentages and Decimals as special parts of Wholes.

In class 6, percentages may be introduced. A link to percentages can be made through the concept of fractions outlined above: they are very special types of fractions. From the analysis to synthesis perspective, percentages are wholes (not necessarily integers) divided by one hundred and multiplied by another number.

Practical examples could be done in terms of an imaginary shop where there is a 20% discount for every 10 loaves of bread bought. Imagine a situation where a person buys 20 loaves of bread at £2 a loaf. What is the total cost?

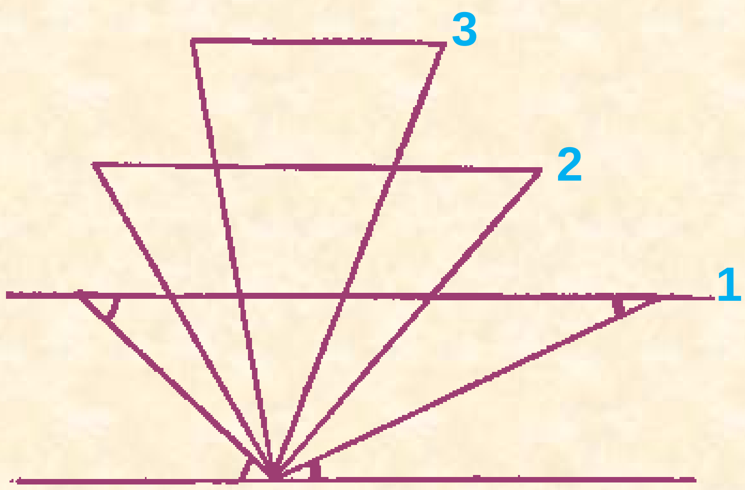
Decimals can be treated in a similar way and then one can proceed to show how they can be represented in their unique fashion: decimals are wholes (not necessarily integers) divided by 10 or multiplied by 0.1.

Geometry and Flexible Imagination

Geometry proper often begins from class 4 onward (see Table 1 near end) although the children will be acquainted with geometric forms through lessons on form drawing given in previous years. The approach to this sub-phase is that of the **Descriptive Method**. The content covered here (but which is not exhaustive) are: **triangles, spheres, circles, ellipses, rhomboids, hexagons, pentagons, spirals, polyhedra**. There are a number of different facets to the Descriptive Method, one of which is the description of **observed** mathematical forms, another is the **imagination** of potential forms:

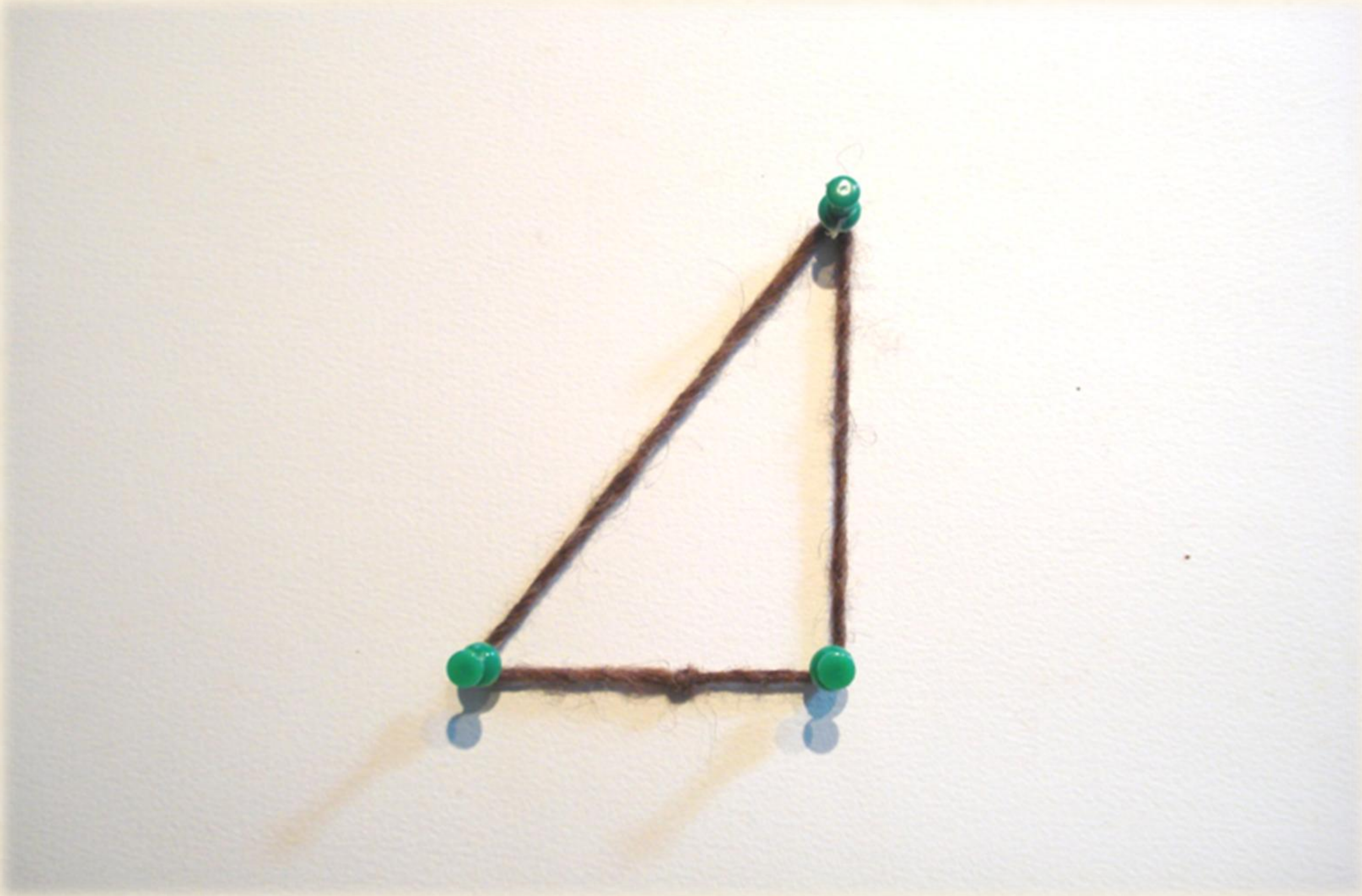
"Geometry can be seen as something that can slowly be brought from a static state into a **living one**. In actuality, we are speaking of something quite general when we say that the sum of the angles of a triangle is 180°. That is true for all triangles, but can we **imagine** a triangle? In our modern way of educating, we do not always attempt to teach children a flexible concept of a triangle. It would be good, however, if we teach our children a flexible concept of a triangle, not simply a dead concept... Thus in moving from three fan-shaped angles lying next to one another, I can form numerous triangles and those triangles thus become **flexible in the imagination**.... It is good to awaken the idea of a triangle of a child in this way, so that an inner flexibility remains and so that they do not gain the idea of a static triangle, but rather that of a flexible shape, one that could just as well be acute as obtuse, or it could be a right triangle (see diagram below)". Steiner, R (1920): The Renewal of Education, Anthroposophic Press, p. 225.

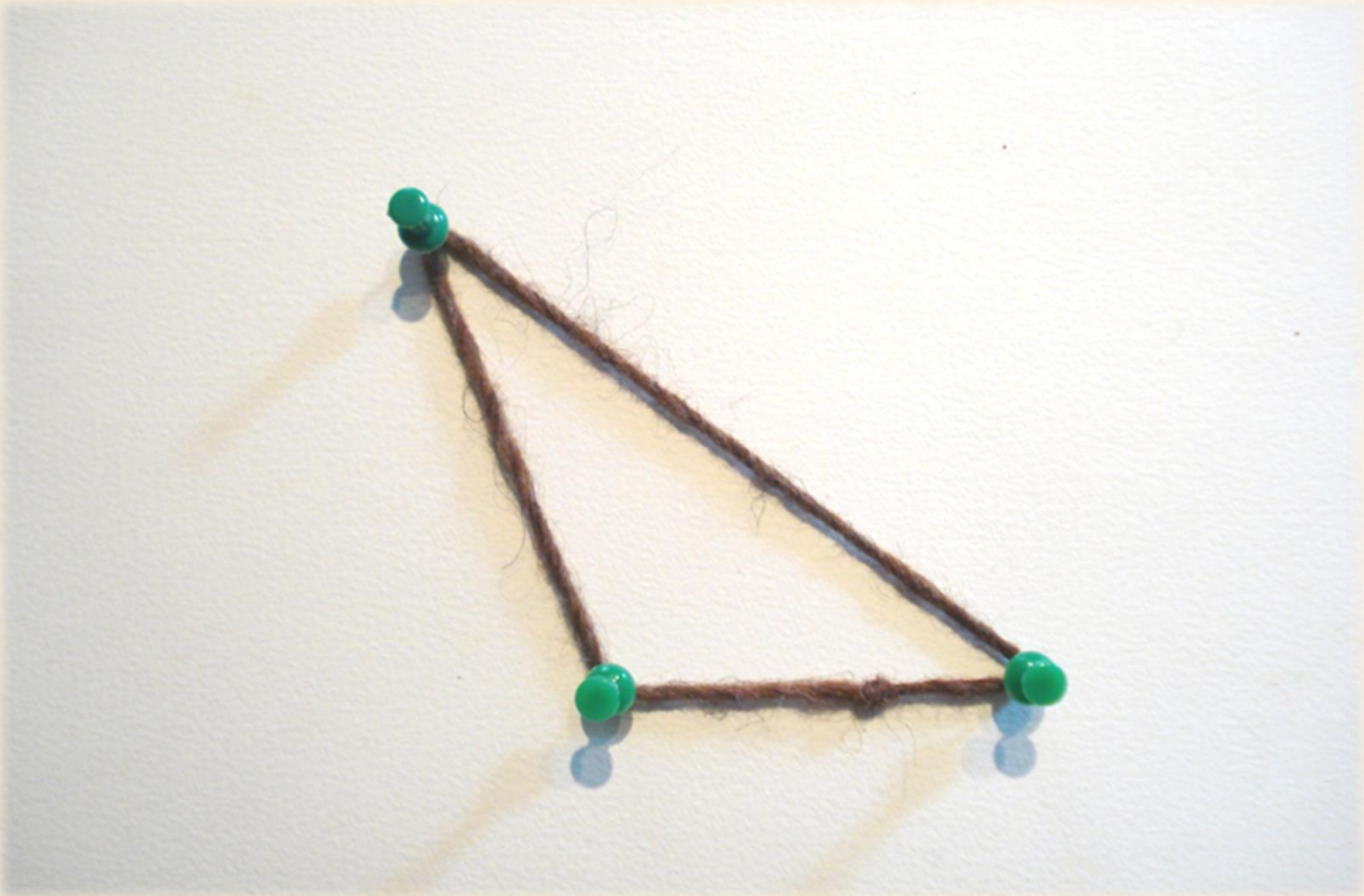
Steiner presented this in the following pictures of triangles (numbers added):

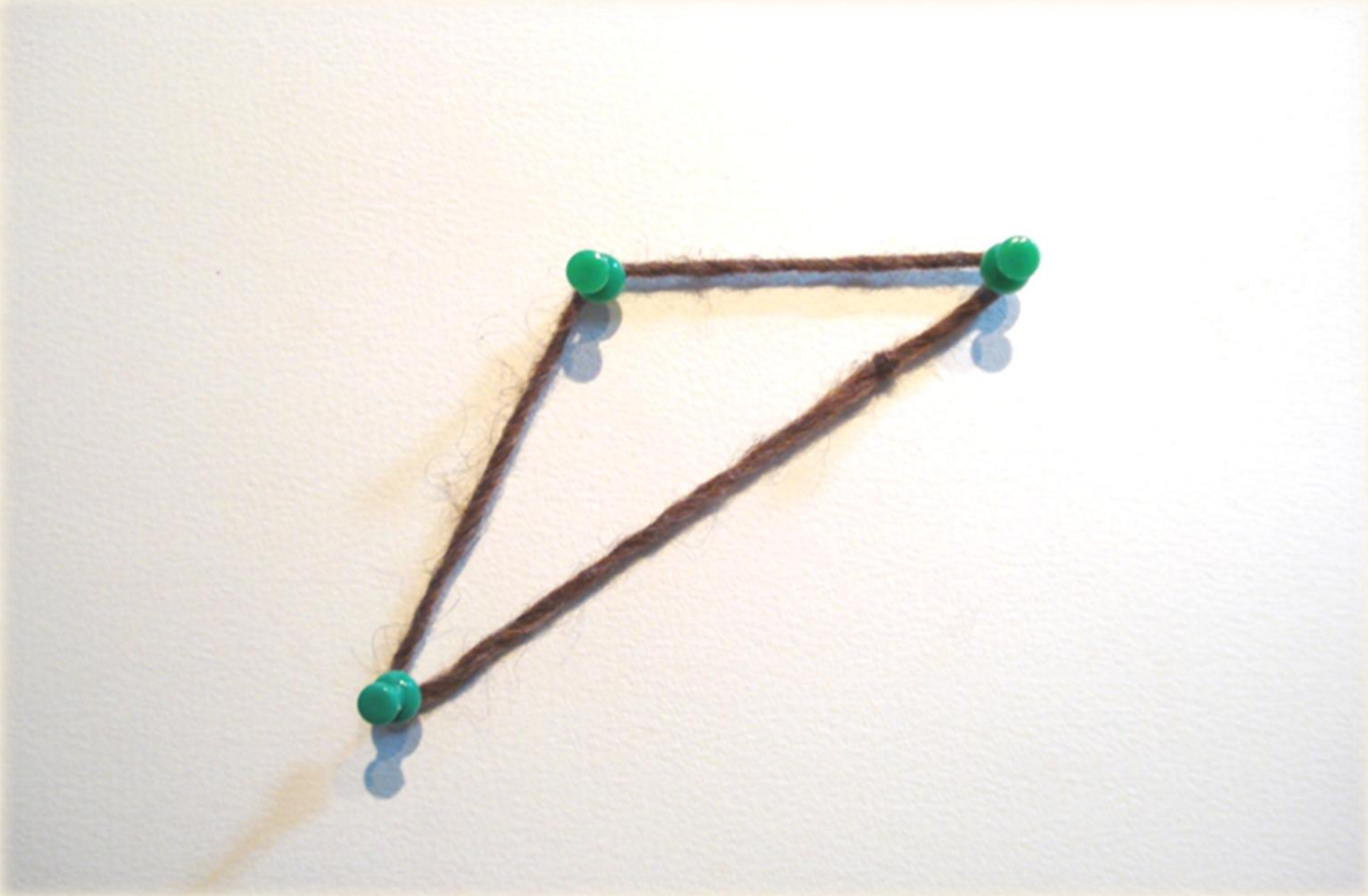


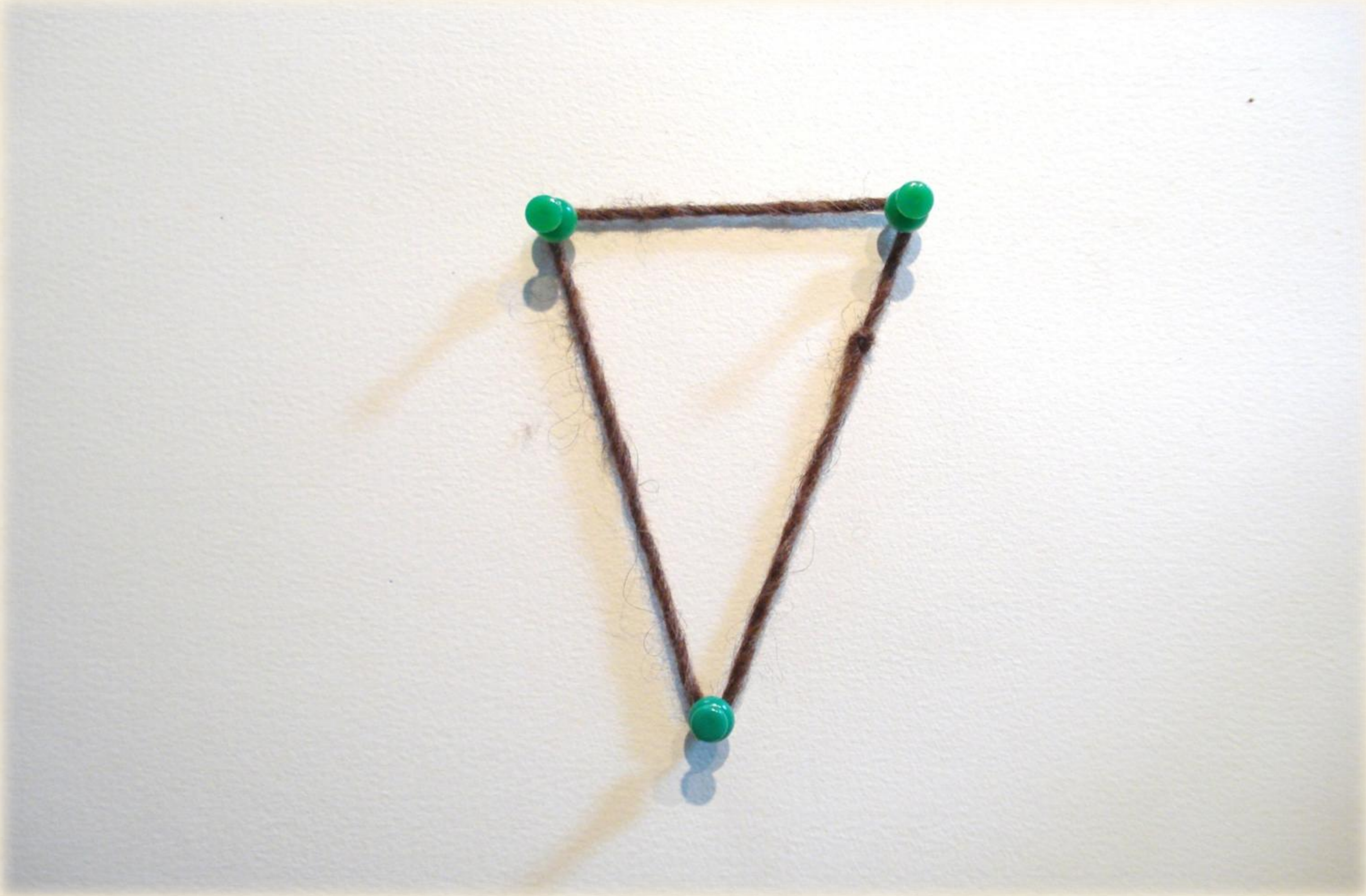
It is possible to picture triangle 1 purely in the mind, then 2, then 3. Imagine picturing them in a continuous sequential transition from 1 to 3. In this process, all in-between triangles are also imagined. One could carry on this exercise such that all triangles between 0° and 180° to 360° can be imagined in sequence. Pedagogically, through an activity such as drawing or observing, it is possible thereafter to imagine **all possible plain triangles** and thereby develop the geometric imagination.

As an aid to this, it might be valuable to "draw", using a piece of wool, a triangle in a slow metamorphic sequence (also try going through the slides quite quickly to see the apparent anti-clockwise motion):





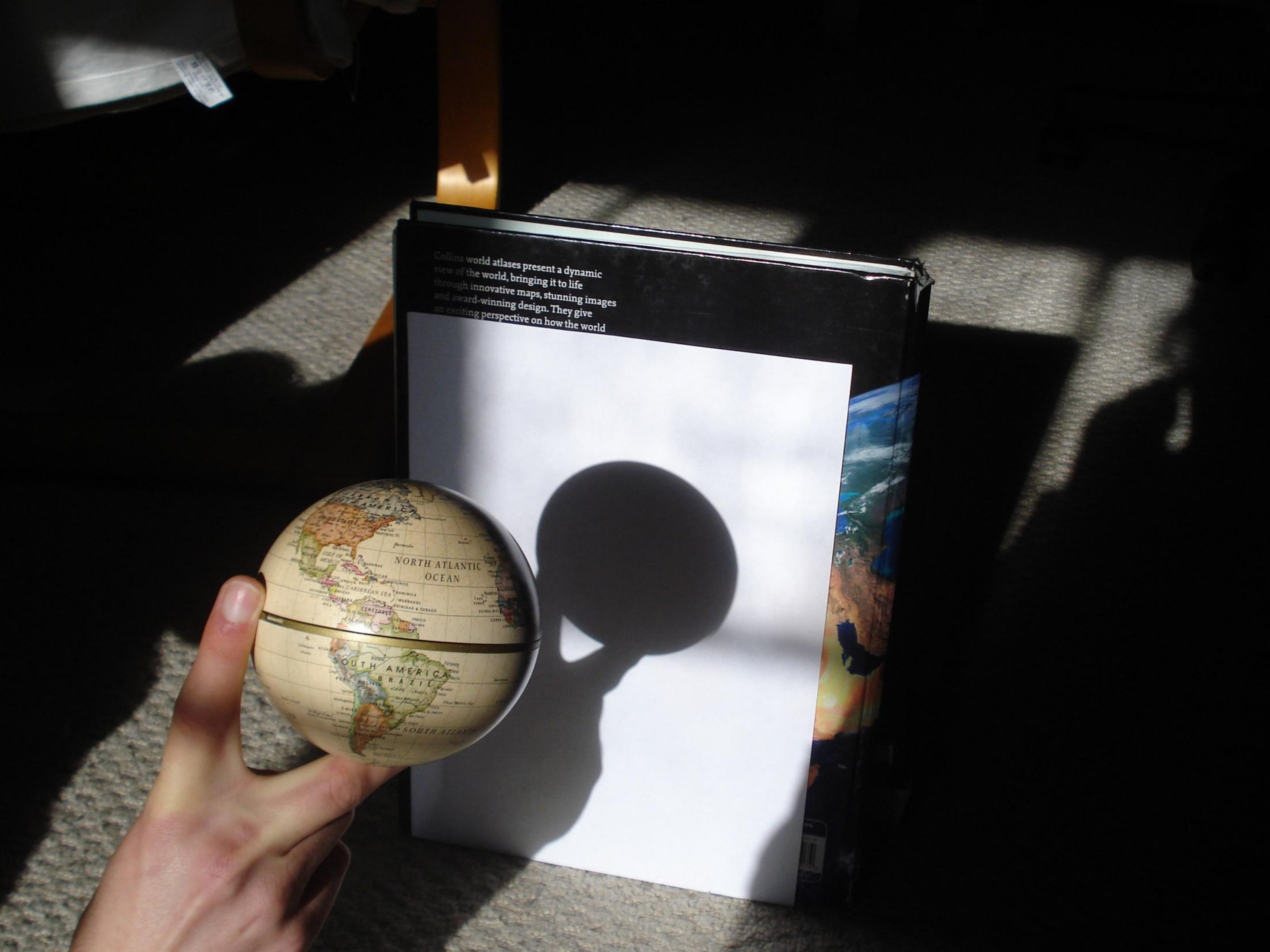




For class 6, Steiner extends this concept to include **movements**, **observation** and **imagination**:

"In the same way if we are to make judgments about the spatial relationship of one being to another, we need to go into the inner aspects of those beings. When properly grasped in a living way, it enables us to develop a spatial feeling in children so that we can use **movements** for the development of a feeling for space. We can do that by having the children **run in different figures**, or having them **observe** how people move in front or behind when passing one another. It is particularly important to make sure that what is observed in this way is also **retained**. This is especially significant for the development of a feeling for space. If I cast a shadow from different objects upon the surface of other objects, I can show how the shadow changes. If children are capable of understanding why, under specific circumstances, the shadow of a sphere has the shape of an ellipse—and this is certainly something that can be understood by a child at the age of nine—this capacity to place themselves in such spatial relationships has a tremendously important effect upon their **capacity to imagine** and upon the **flexibility of their imaginations**." Steiner, R (1920): The Renewal of Education, Anthroposophic Press, p. 227/8. (My bold)

The following series of pictures show examples of a sphere casting shadows which gradually change from the circular to the elliptic: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view of the world, bringing it to life
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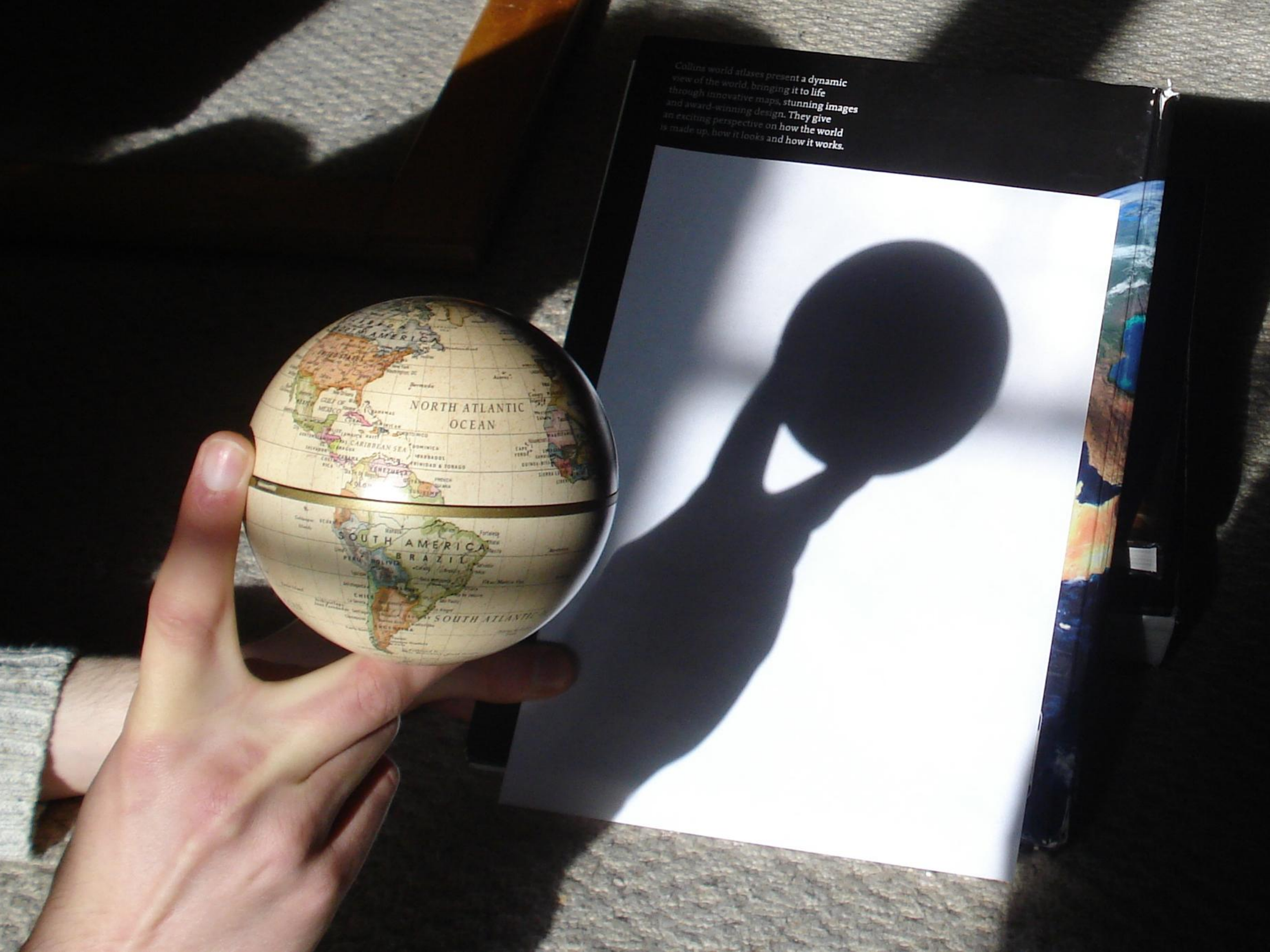
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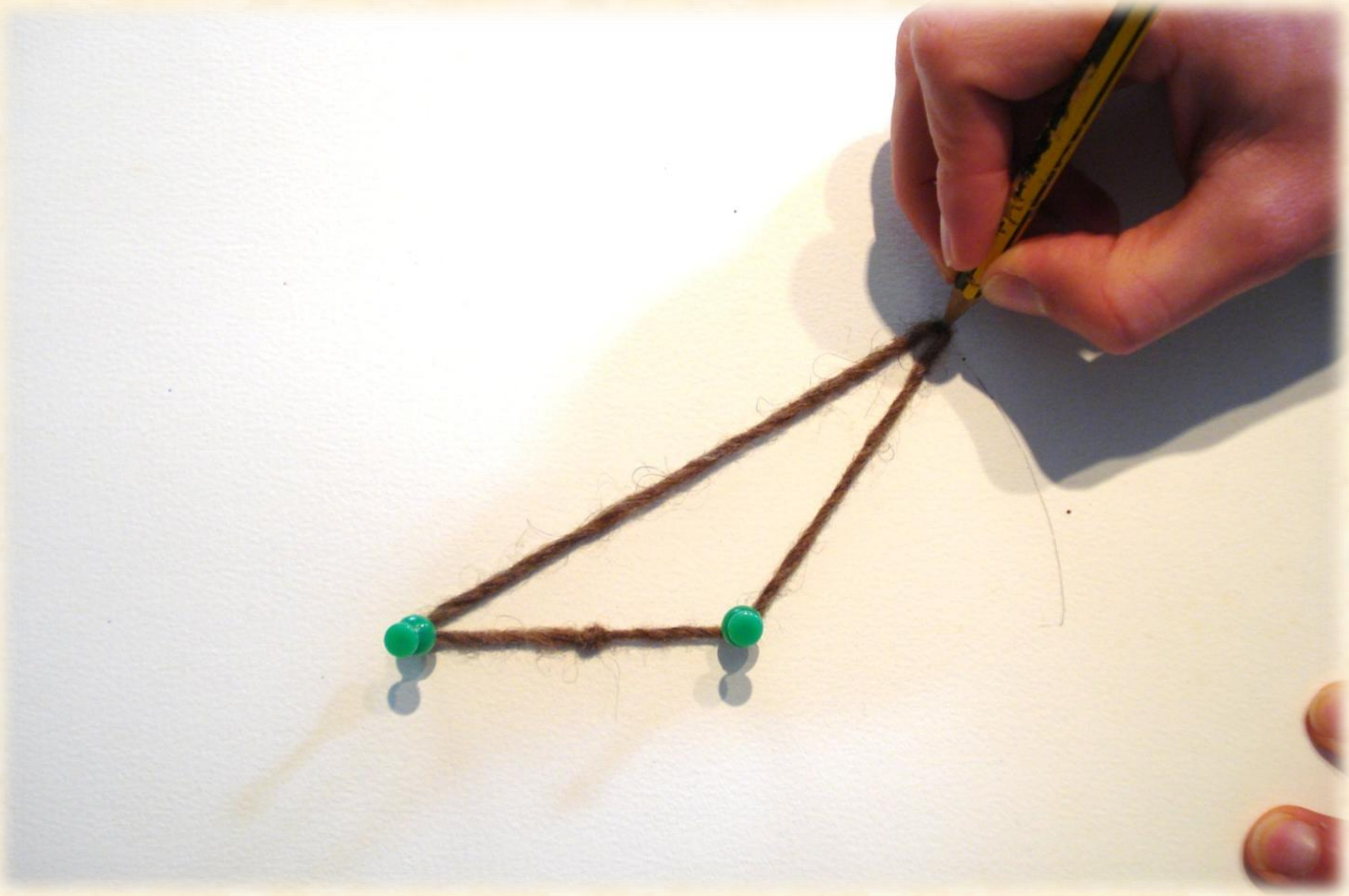
OCEAN
SOUTH AMERICA
BRAZIL


TIMES
CONCISE
ALGERIA
LIBYA
NIGER
CHAD
NIGERIA

You could also try drawing the ellipse, the next picture gives an indication of how. Attempt to see what happens when the two pins are placed closer together: it converges to a circle!



Geometry in Nature and Art as Imagination Development

As was described earlier, Steiner intended that geometry may be done in such a way that it encourages the development of the pupil's imagination. Part of this is the engagement with real mathematical forms that can be actually **observed** in nature and art. These observations can then become a rich **memory** source within the children and which can be drawn on later when mathematics becomes more formal and subject to the method of proof. In this sub phase, the approach to geometry is descriptive and engaging of the will and the artistic sense.

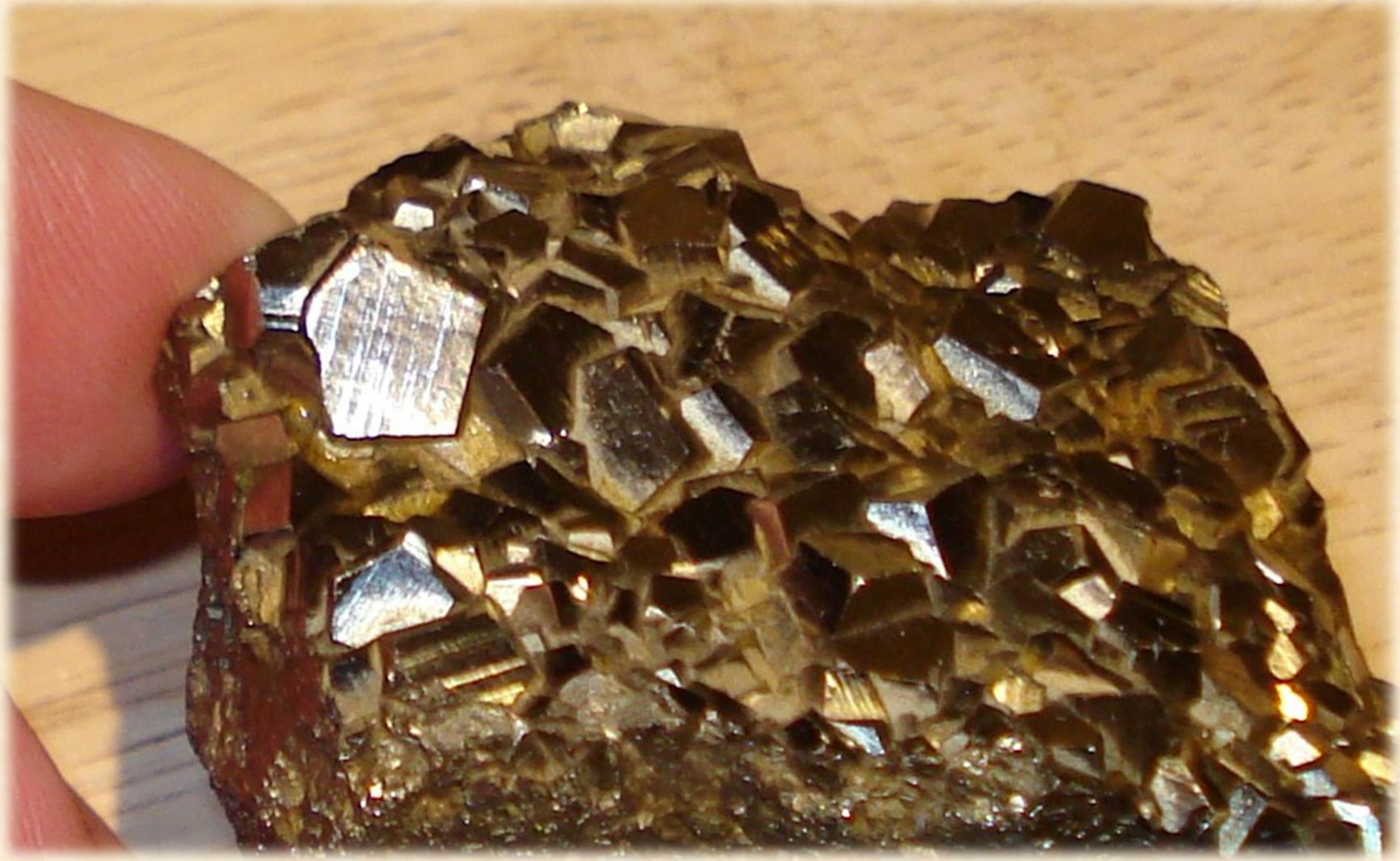
In this section, I want to introduce some aspects of Steiner's philosophy of geometry education in light of the above indications and as correlated to the mathematics curriculum. In the following, it also has to be kept in mind that mathematical forms that are **observed** are never perfect in comparison with the pure mathematical form. This is an idea around which teachers will need to navigate in their working with the children. Moreover, in actual practice, the teacher will transform what can be observed into an artistic creation of their own devising. From out of this, the pupils would be engaged in the observing, drawing, painting and modelling of geometric forms which can be formulated mathematically. The next few images provide an **observational** basis for this:

Geometry in the Mineral Realm

Calcite Rhomboid

Quartz Hexagon

Pyrite Pentagon

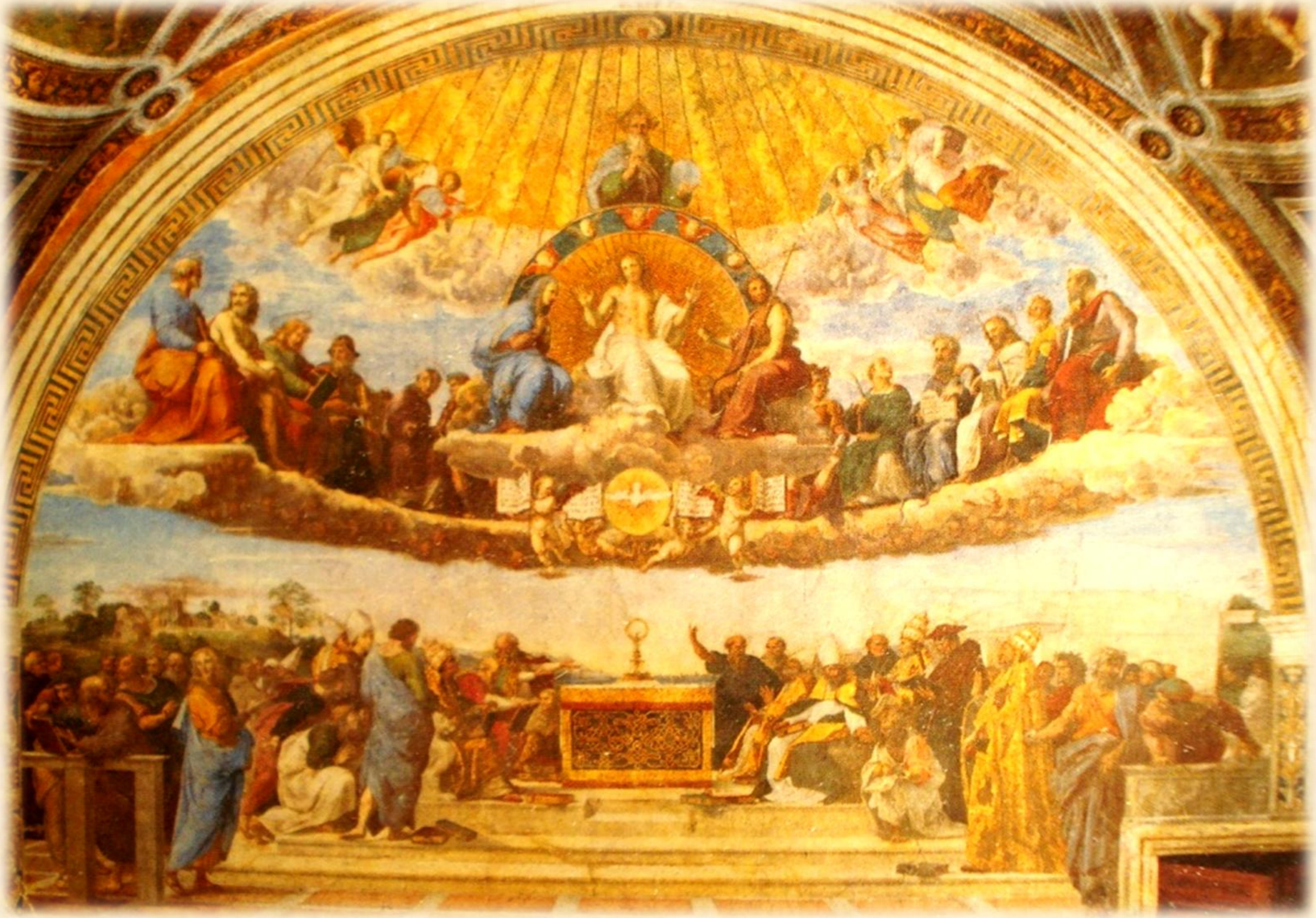


Geometry in the Living Realm & Culture: Spirals

Geometry in Art and Architecture:

The School of Athens by Raphael (1483 – 1520). Four semi-circles in the vertical plane receding in perspective.

The Disputations by Raphael (1483 – 1520).
Semi-circles and circles: how many, which plane?



The Eden Project, Cornwall. Parts of Spheres. Hexagons within spheres.

Circles for the Spiritual: Truro Cathedral.

Willing the Circle

In Steiner Waldorf schools, the pupils would frequently be engaged in creating mathematical forms through drawing or painting. Rather than just **observing** and **thinking** about mathematics, this activates the **will** and the **artistic / feeling** capacities. The picture below was created using wax crayons.

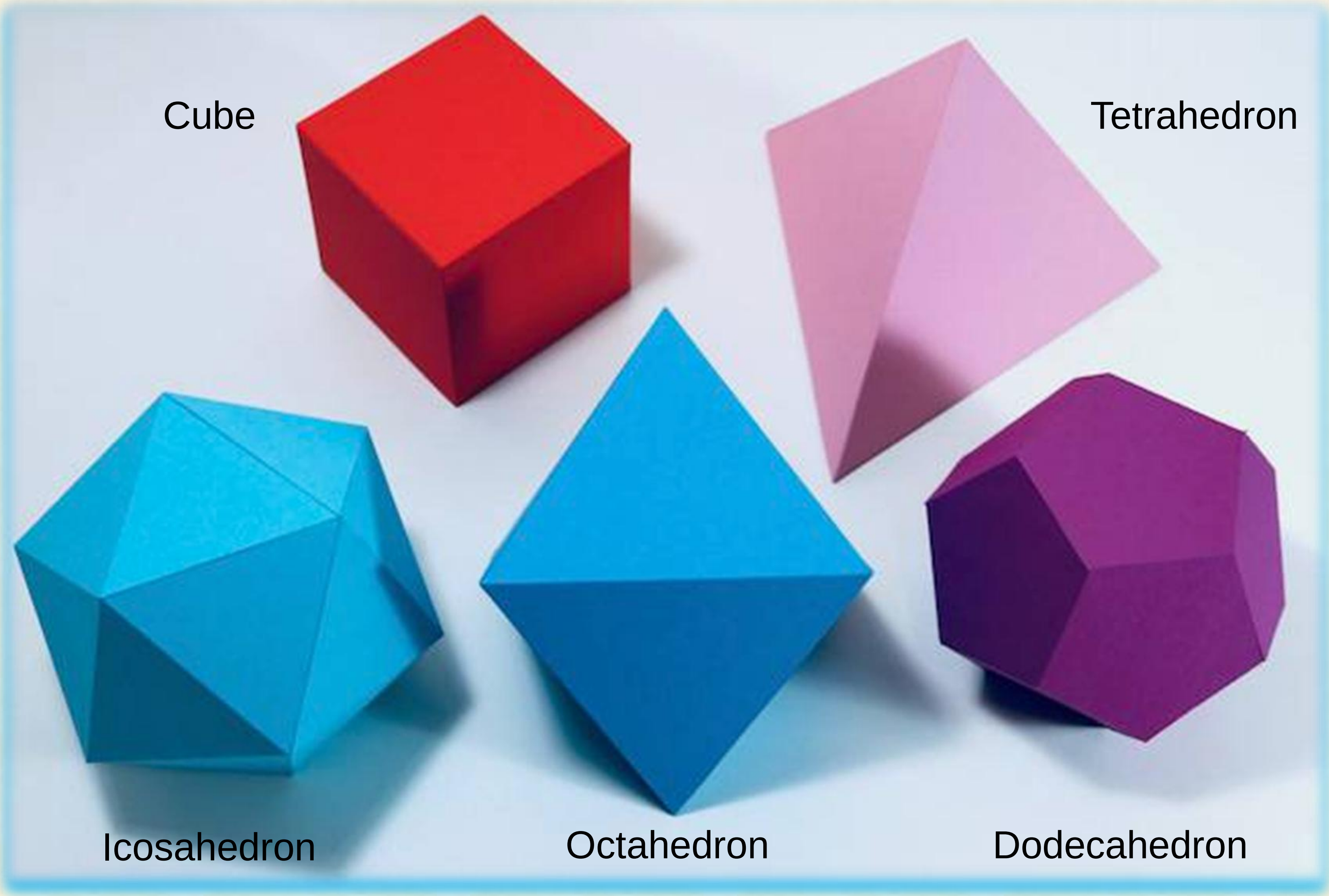
Whole Body Geometry - The Three Dimensions of Space

Steiner was keen that pupils learn about the three dimensions of space through their whole body experience. Again, as a part of his holistic philosophy of learning, rather than the three dimensions being seen as abstract lines or planes in space, he considered it more valuable for pupils to understand these as related to the whole human body experience:

"In later years, the child will be introduced to many different subjects, such as Geometry… three-dimensional space wants to be experienced as reality. This does happen in a young child, although unconsciously, at the crawling stage when, losing its balance time and again, it will eventually learn to acquire the upright position and achieve equilibrium in the world. Here we have a case of actual experience of three-dimensional space. This is not merely a question of drawing three lines in space, because one of these three dimensions is identical with the human upright position (which we can test by no longer assuming it—that is, by lying horizontally or sleeping). This upright position signals the most fundamental difference between the human being and the animal, because, unlike the human backbone, the animal's spinal column runs parallel to Earth's surface. We experience the second dimension unconsciously every time we stretch our arms sideways. The third dimension moves from our front toward the back." Steiner, R (1923): The Child's Changing Consciousness, Anthroposophic Press, p. 17/18.

From out of this, it is then possible to form a lived experience of the typical forms found in geometry, first of all in the three dimensions of bodily experience. Eurythmy classes can be an aid to this as can Bothmer gymnastics. This can then also be explored in the experience of objects such as polyhedra and the platonic solids as represented in the next slide. Pupils may be given the opportunity to experience their own three dimensionality in relationship to these forms through artistic activities such as clay modelling and Eurythmy. By engaging their physical bodies in modelling, a deeper experience is engendered of the speciality of such forms in three dimensional space:

The Five Platonic Solids



Sub Phase 3 (Classes 6-8)

In this phase a new step is taken in terms of the mathematical method, the **method of proof (see Table 1)**. There are two aspects of this: the **algebraic** and the **pictorial**. Clearly, the first of these is closely connected to the first teachings of algebra in Steiner Schools and the second is particularly valid for geometry and the application of algebra to it. As we have seen, the latter of these has been approached from a descriptive method perspective, it now becomes a question of proving geometric relationships through algebra. As we have seen in module 3, this ties in with the first preview of conceptual thinking in child development which has its inception in the late 12th year of a child's life.

Geometry in Steiner schools goes through a number of stages as described previously.

"It is difficult to explain to a child what an angle actually is. Can you make up a method for doing this? Perhaps you remember how difficult it was for you to be clear about it—aside from the fact that there may be some of you who do not yet know what an angle really is." Steiner, R (1919): Discussions with Teachers, Anthroposophic Press, p. 149.

"But if by tomorrow you would consider the whole subject on a somewhat different basis, perhaps you will find it beneficial to introduce the children to a clear concept of area as such first, and then the size of the area. The children know the shape of a square, and now you want to show it to them as a *surface that* could be larger or smaller." Steiner, R (1919): Discussions with Teachers, Anthroposophic Press, p. 150.

"What is the right way to proceed, and at what age, in order to actually discover what a surface really is, and that it is obtained by multiplying length by breadth. How can you manage to awaken this concept of a surface in the child? This depends on when you begin teaching children about surface areas. It doesn't make sense to teach them about surface areas until after you teach them some algebra. **The answer, therefore, is to wait for lessons on surface areas until after we deal with algebra**." Steiner, R (1919): Discussions with Teachers, Anthroposophic Press, p. 151/2. (My bold).

From Arithmetic to Algebra

As can be seen, the teaching of geometry is intended to follow that of algebra. We will consider this next. The Wholes and Parts relationship is not only relevant to the context of arithmetic, Steiner also applies this to algebra. In his lecture series “Discussions with Teachers” he shows how the transitions from arithmetic to algebra can be made, and also how the whole sequence of the arithmetic and algebraic processes can be derived. **It would be worthwhile at this point to read the Fourteenth Discussion from “Discussions with Teachers”**. In the text, Steiner shows how from an addition equation multiplication can be developed and from a subtraction equation division can be evolved. In both cases, the whole, i.e. the **Total** is expressed first as a number then as a letter, such as:

Whole	Parts
9 =	3 + 3 + 3
S3 =	a + a + a.........3 times = 3a (multiplication from addition)
Sn =	a + a + a.........n times = na (multiplication from addition)
Rn =	a – b – b........ n times; b = a/n for no remainder (division from subtraction)

For Steiner then, the whole is always the starting point for the mathematical process, which later on is reversed: ***first* from the whole to the parts, *then* from the parts to the whole**. He concludes:

"In this way multiplication can easily be developed and understood from addition, and you thus make the transition from actual numbers to **algebraic** quantities. In the same way you can derive division from subtraction… From addition, therefore, you develop multiplication, and from multiplication, rise to a higher power. And then from subtraction you develop division, and from division, find roots:

Addition	Subtraction
Multiplication	Division
Raising to a higher power	Finding roots

You should not proceed to raising to a higher power and finding roots until after you have begun algebra (**between the eleventh and twelfth years**), because, with roots, raising to a power of an algebraic equation of more than one term (polynomial) plays a role. In this connection you should also deal with figuring gross, net, taxes, and packing charges." Steiner, R (1919): Discussions with Teachers, Anthroposophic Press, pp. 159-161. (My bold and modified table).

Algebraically, as we know, these mathematical processes can be represented as:

Polar Opposites

Addition:

A + B

Subtraction:

A - B

Multiplication:

A x B

Division:

A / B

Raising to a higher power:

A^n

For example

A^3

Finding roots:

$\sqrt[n]{A}$ or $A^{1/n}$

For example

$\sqrt[3]{A}$ or $A^{1/3}$

Geometry

Angles as Part of Wholes

As pointed out earlier, Steiner was aware of the difficulty some children can have with the idea of angles in geometry. His approach to this involved a **qualitative** approach before a quantitative one:

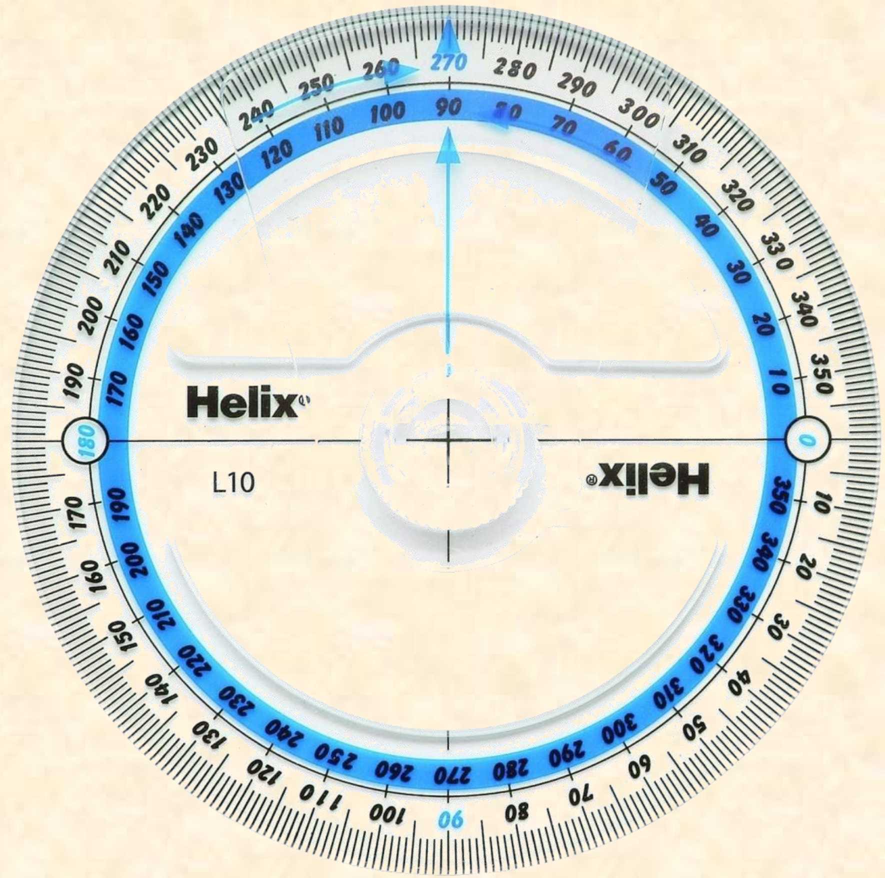
"It is difficult to explain to a child what an angle actually is. Can you make up a method for doing this? Perhaps you remember how difficult it was for you to be clear about it—aside from the fact that there may be some of you who do not yet know what an angle really is. You can explain to the children what a larger or smaller angle is by drawing angles, first with longer arms and then with shorter arms. Now which angle is the larger? They are exactly the same size! Steiner, R (1919): Discussions with Teachers, Anthroposophic Press, p. 149

"Then have two of the children walk from a certain point simultaneously, two times, and show them that the first time they walked they made a larger angle, and the second time a smaller one. When they walked making the smaller angle their paths were closer together, with the larger angle further apart. This can also be shown with an elbow movement.

It's good to arrive at a view of larger and smaller angles before beginning to measure angles in degrees." Steiner, R (1919): Discussions with Teachers, Anthroposophic Press, p. 149/50.

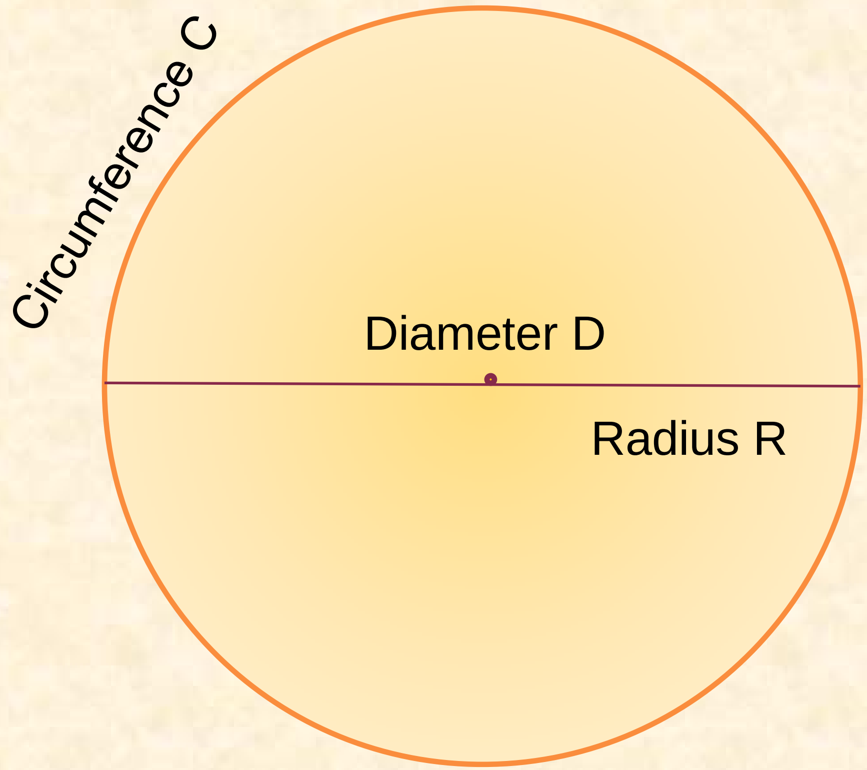
A step further may be to ask the children to imagine a cake which is cut into equal slices (one could use an actual cake!). From this a connection can be made to a very large cake, big enough for a whole school which has 360 children each with the same sized slice. This could be related to a circle which has 360 slices. A transition could then be made to a compass or a protractor which are also divided into 360 "slices", what adults call 360°.

In this way, a step is made towards a quantitative understanding of angles; slices become angles: the circle is the whole, the parts are the 360°:



Π – A Whole Relates to a Part of itself

It may be around this point that a teacher may chose to introduce Π. Given a circle, as is well known, there is a special ratio of the diameter to the circumference. Consider the circle and divide it by its diameter:



$$\Pi = \frac{C}{D}$$

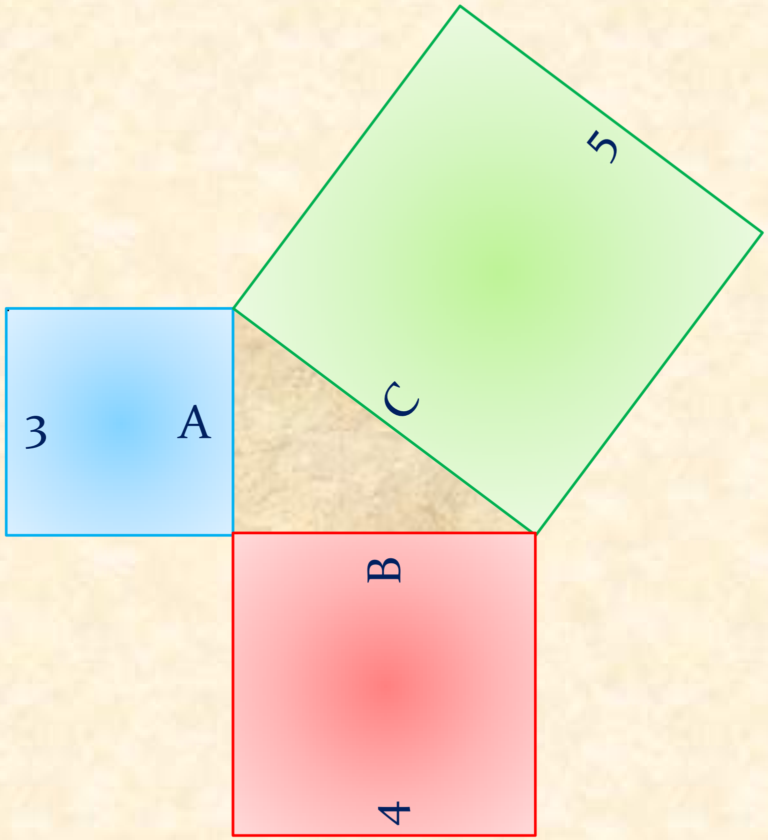
Or

$$\Pi = \frac{C}{2R}$$

This special number Π is always 3.1415…, no matter what size the circle. Holistically, it is the constant ratio of a whole to its symmetrical cross section R. In other words, it is how a particular kind of whole relates to a part of itself.

Geometry and Pythagoras' theorem

Following on from Algebra, Geometry is further developed in the third sub-phase, after the nature of squares and roots has been covered in algebra. This particularly the case with right angled triangles and the proof of Pythagoras' theorem, for example:



$$C^2 = A^2 + B^2$$

Or

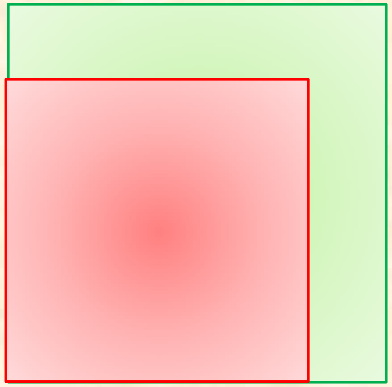
$$C = \sqrt{A^2 + B^2}$$

Or

$$C = (A^2 + B^2)^{1/2}$$

The first kind of **proof** that Steiner speaks of is visual proof. One way of approaching this would be to cut out the shapes in accord with the above picture assuming the numbers to represent centimetres. **You might like to try this as an exercise**.

Take the 4/4 square and lay it on top of the 5/5 square. This should give you this:



Then cut up the 3/3 square into the appropriate shapes to give this:

It fits perfectly.

An algebraic approach could then follow on from this. As is well know this is given by:

$$C^2 = A^2 + B^2$$

Or

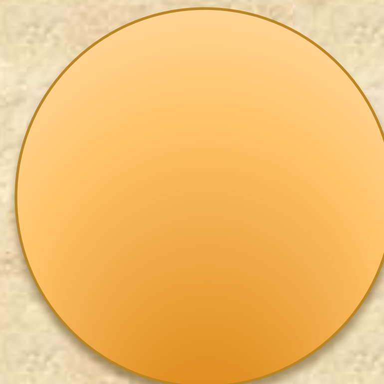
$$C = \sqrt{A^2 + B^2}$$

Or

$$C = (A^2 + B^2)^{1/2}$$

As this is one of the mathematical goals for the end of the lower school, it is worthwhile considering the pre-requisites for its understanding. Clearly, an understanding of addition is needed as are powers, roots, areas and the use of equalities, brackets and signs. Essentially, most of the mathematics curriculum up to this point is needed for an understanding of Pythagoras's theorem: it is existentially dependent on these. Evidently, then, the children will need to know these from prior teaching.

One could also connect the algebra learning with what has been learned from geometry. In each case, the **whole** on the Left Hand Side of the equation is seen as the origin of the **parts** on the Right Hand Side (you may need to revise your maths here):



Ellipse equation:

$$1 = \frac{X^2}{A^2} + \frac{Y^2}{B^2}$$

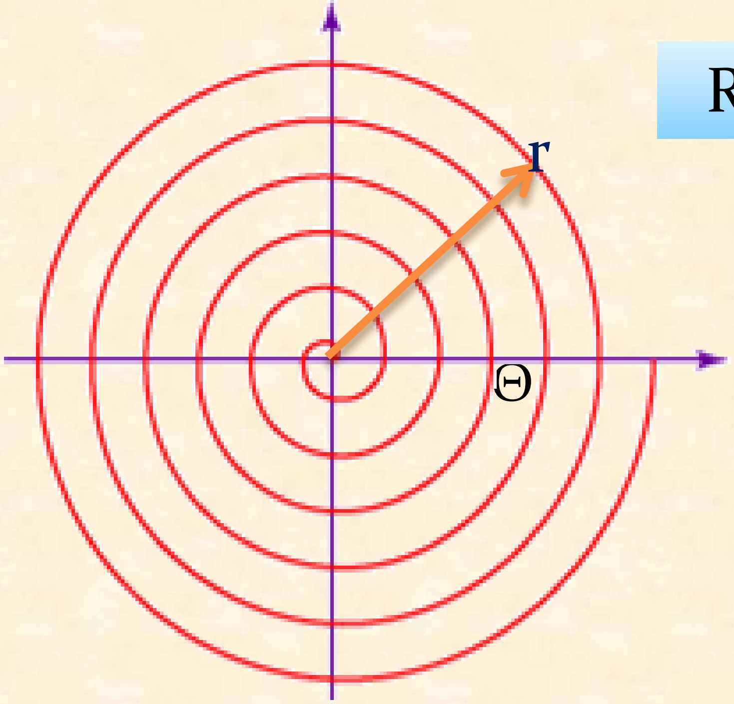
Circle Equation: when A = B, the ellipse equation reduces to a circle equation:

$$A^2 = X^2 + Y^2$$

Spiral Geometry and its Algebra

We could also apply the algebra to spirals as in our spiral snail shown earlier. Archimedes used geometry to study the curve that bears his name. In modern notation it is given by the equation **r = aθ**, in which a is a constant, r is the length of the radius from the centre, or beginning, of the spiral, and θ is the angular position (amount of rotation) of the radius.

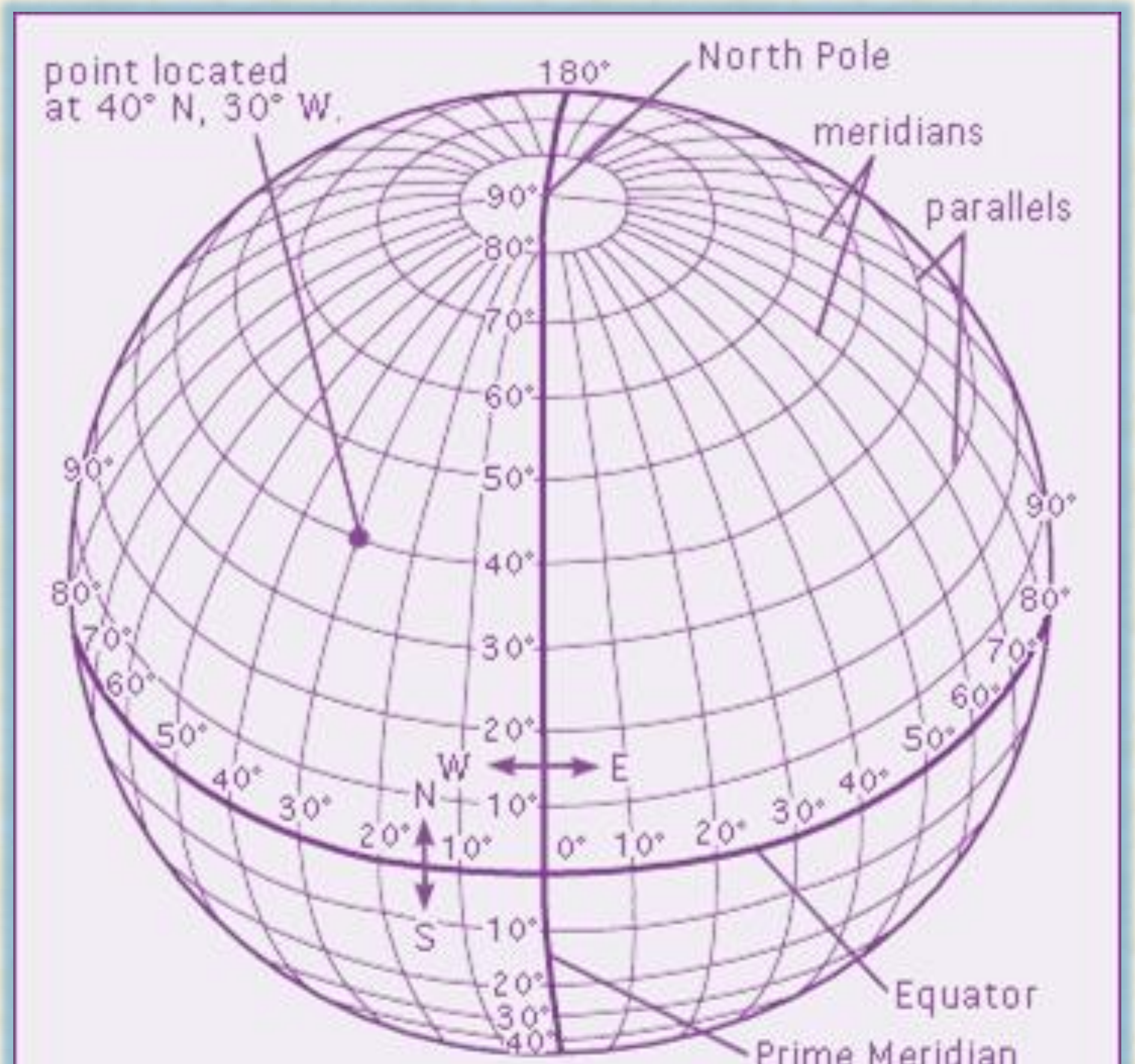
Radius r = aΘ



Spherical Geometry as World Geometry

It isn't usual that spherical geometry is taught in class 8, normally, spherical geometry would be covered in class 11, but an **elementary** introduction would be quite consistent with the global trends within the other parts of the curriculum and the theory of development of the 14 year old adolescent consciousness advocated by Steiner. So this section is more in the way of a potentiality rather than an actuality in Steiner School settings. In addition, Steiner made the recommendation that geometry be taught "as far as possible", so this section is a brief attempt to consider what this could mean.

Consider the following mathematical image of the World, which is a part of a subject called "geodesy":


point located at 40° N, 30° W.
180°
North Pole
meridians
parallels
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
W
E
N
S
Equator
Prime Meridian

As we have seen in both human geography and biogeography the teaching in Steiner / Waldorf settings tends towards the World image in union with the image of the Human being as well as being valid in astronomy:

"You should emphasize spherical trigonometry and how it is used in astronomy and geodesy [the scientific study of the shape and size of the Earth] in a way **appropriate to their age**, so that they have a general understanding of it." Steiner, R (1919-24): Faculty Meetings with Rudolf Steiner, Anthroposophic Press, p. 744. [My addition in brackets and bold]

However, it might be valuable to the students of class 8 to make a some **elementary** inroads into the mathematics of the globe and universe using a practical, pictorial and basic algebraic approach.

A number of other developmental perspectives also arise:

Speaking of the method for understanding spherical geometry Steiner states:

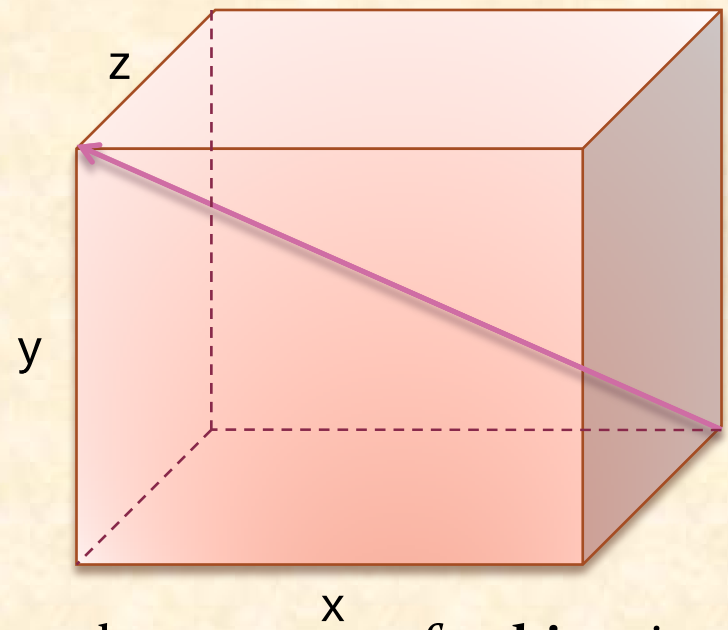
"Now that the class is ready for spherical trigonometry, you will need to move from trigonometry to developing the concept of the sphere **qualitatively**, that is, without starting computations. Instead of drawing on a plane, they need to begin drawing on a sphere, so that they get an idea of what a spherical triangle is, that is, how a triangle lies upon a sphere. You need to make that visible for the children, then go on to show them how the sum of the angles is not equal to 180°, but is larger. They need to really understand triangles on a sphere, with their curved lines, and then begin the computations. In geometry, the computation is only the interpretation of the sphere. I do not want you to **begin** by considering the sphere from its midpoint, but from the curvature of the surfaces." Steiner, R (1919-24): Faculty Meetings with Rudolf Steiner, Anthroposophic Press, p. 746.

Clearly, even in class 11, Steiner wanted the approach to be initially a qualitative one, so this would be even more the case for class 8. He also wanted the study to begin on the surface of the sphere, rather than the middle, this would come afterwards.

Firstly, the mathematics of spherical surfaces show different rules from those of flat surfaces, such as those presupposed by the laws of the Pythagorean triangle. In the case of plane surfaces, it is known that the sum of the angles of a right angled triangle equals 180°. This is not the case for spherical triangles! In the above picture, the two meridians of longitude are separated by an angle of 90° and both lines of longitude fall perpendicular to the Equator (the only great circle of latitude). Each angle in this particular spherical triangle equals 90°, and the sum of all three add up to 270°! A qualitative approach to this might involve a globe and some string to outline the triangles wanted.

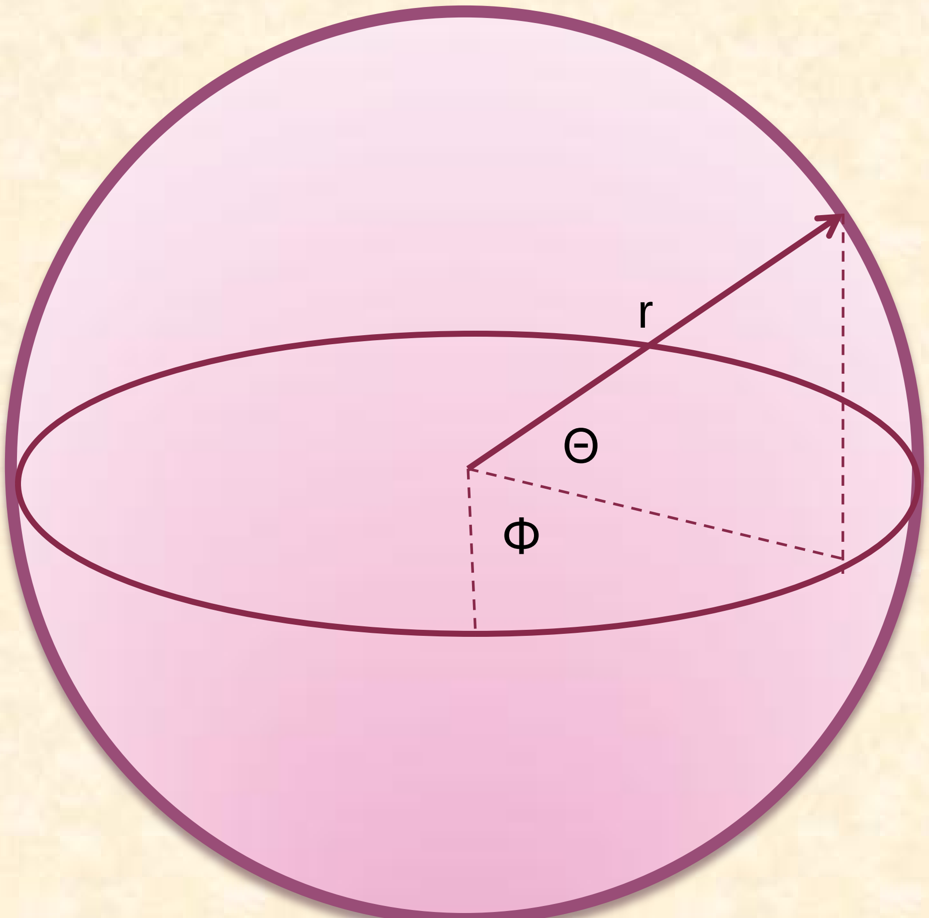
Such considerations enables the students to transcend the rules and principles which they are accustomed to: truth becomes contextual rather than absolute, but not random or subjective. An important lesson to learn for the young adult.

Another aspect to this is that spherical geometry brings mathematics closer to **human** reality. All human movement is grounded in the fact that our joints are of a number of different types of spheroids (albeit approximately) and this conditions how we move: our motion through space is derived from limited types of spherical and circular movement. For instance, instead of Cartesian coordinates of x, y, z:

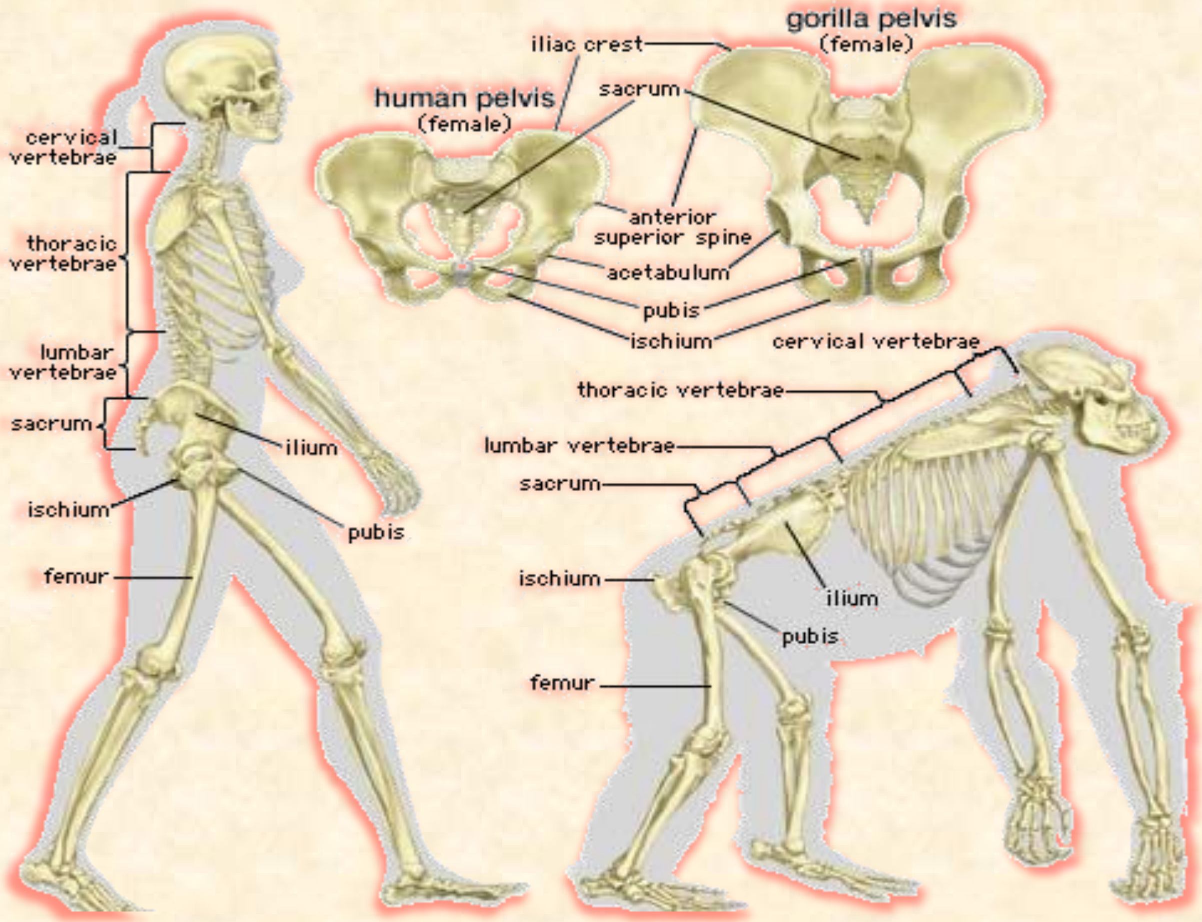


Which is based on the geometry of **cubic** *mineral* forms, we have a geometry derived from an understanding of **spherical** *human* forms.

Steiner recommended the study begin on the surface of a sphere. Once this has been done, the inner dimensions can come into consideration. Below, we have spherical coordinates of r, Θ, Φ, which describe the movement of our individual limbs and the summed outcome of these in our movement through space:

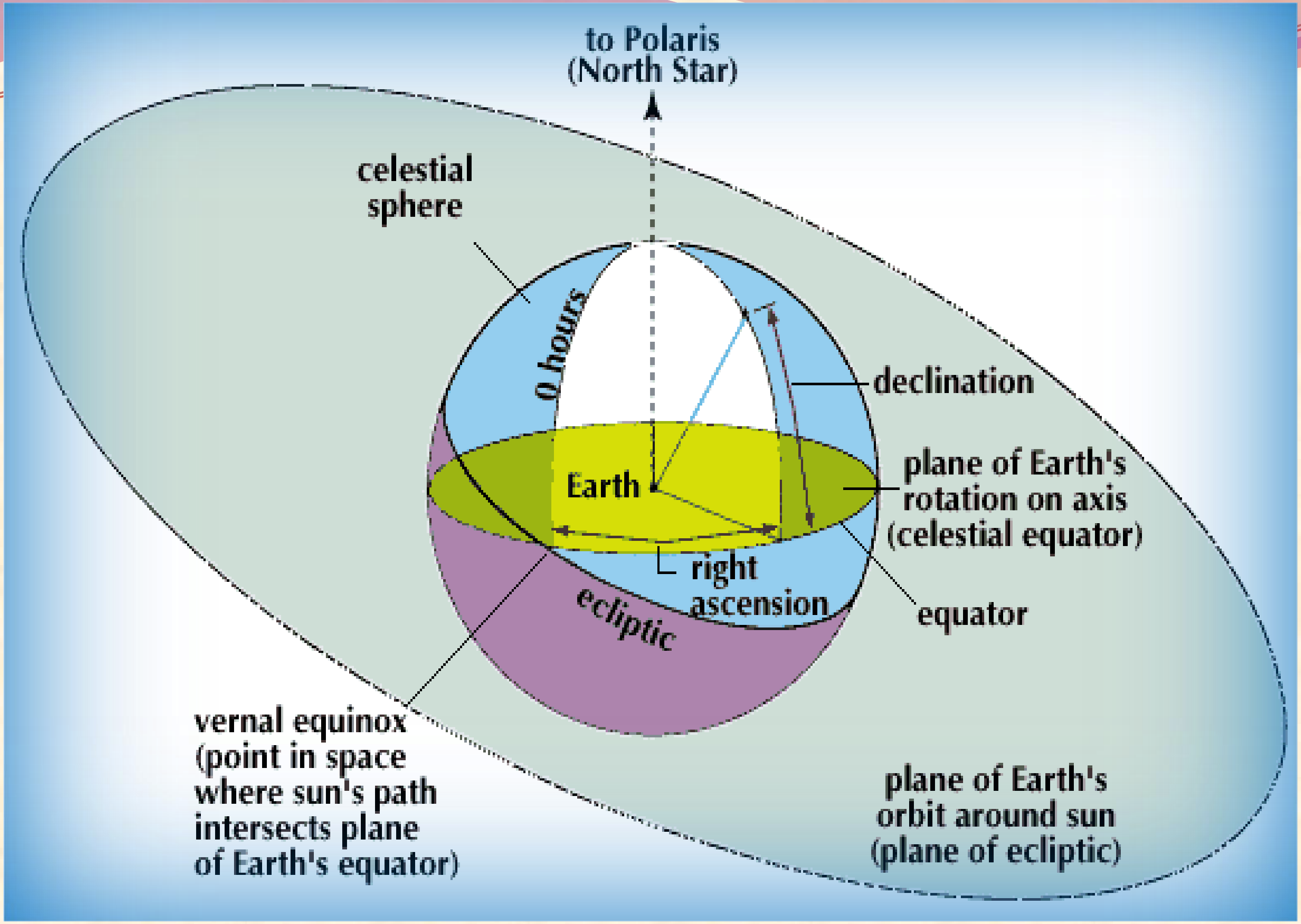


Take a look at the joints in this picture to see if you can indentify the quasi “spherical” forms in our limbs and which condition human and animal movement. Make some observations of how you, as well as other people, walk. Pay special attention to the movement of individual joints in connection with movement as a whole.



From this, a human based mathematics can be seen to be extendable to the whole world and indeed to the universe. The first in terms of distances between places of the Earth:

The second being concerned with our place in the universe:



This is one example of how mathematics can become human and seen to be a part of the World and Universe; a mirror for the changing stage of development for the young teenager: **from the Self to the World.**

Table 1 Summary of the Mathematics Curriculum Lower School from “Discussions with Teachers”

Sub-Phase 1: Imaginative Anthropomorphisms			Sub-Phase 2: Descriptive Method			Sub-Phase 3: Proof as Method	
Class 1 **6-7 yrs**	**Class 2** **7-8 yrs**	**Class 3** **8-9 yrs**	**Class 4** **9-10 yrs**	**Class 5** **10-11 yrs**	**Class 6** **11-12 yrs**	**Class 7** **12-13 yrs**	**Class 8** **13-14 yrs**
Introduction to counting and numbers. Deriving the parts from the whole. The four arithmetic processes: from whole to parts and reverse. Learning the times tables by heart.	Continuing from class 1 with a higher range of numbers. Introducing unknowns and their calculation.	More complicated numbers. More complicated arithmetic processes.	Developing earlier classes, introducing fractions and decimals. Introducing geometry: circles, ellipses and other forms through observation, painting and drawing. Three dimensional forms through modelling real objects. Develop mathematical awareness of forms in nature through observation and drawing, etc.	Develop further fractions and decimals. Develop independent ability to do calculations in topics introduced.	Calculate percentages, interest, discounts. Geometry: squares, triangles, circles. Three-dimensional forms. Mathematical Projections and shadows, etc. Mathematics of the connection between the technical and the beautiful.	Introduction to algebra. Powers and roots. Positive and negative numbers. Applying maths to real life situations. Mathematics of object penetrations and interjections. Further development of the connection between the technical and the beautiful.	Continue algebra, powers, roots, positive and negative numbers, practical applications. Calculating areas and volumes. Geometric Loci. Geometry as far as possible.

Exercises

1) For class 2 or 3, construct a lesson, using the principles from module 5 (L1a to L1d), around the following verse:

As I was going to St Ives,
I met a man with seven wives.
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits.
Kits, cats, sacks and wives,
How many were going to St Ives?

Let us ignore the trick solution to this in which there is only one person going to St Ives. Find a way to introduce this to the children in poetic or song form. From out of this devise some practical activities for the pupils to engage in. Obviously you will need to have build up their maths to the point where they can actually carry out the calculation. You may wish to do the calculation in terms of addition and multiplication, or for older children in sub phase 3 you may wish to use powers. For the latter case, consider how you could introduce this algebraically for any number of wives but each with the same number of sacks, cats, and kits.

2) For class 4, explore some possibilities of shadow projections using a number of different shapes. Obtain a torch or a candle and shine onto a range of different shapes and note the transformations in the form of the shadows as you move the torch around. Think about how you would describe these mathematically to the children.

Consider also a sundial, try to make one and if necessary use a torch to represent the Sun during the course of a day and year. Try to use mathematics to describe what you see. You might find it helpful to look at the history of sundials and what you can learn about the Earth from them as well as Time.

3) For a class 8 main lesson, obtain some clay or similar material and model the platonic solids. Begin with moulding a sphere and then gradually transform it into the platonic solids one after the other. Then create each form individually and place them next to each other in a sequence based on the closest similarity of form. Consider how you could represent this mathematically in as simple a way as possible. Think about how you may create a main lesson from out of this using the principles introduced in module 5.

4) This is a picture of one of the buildings of the South Devon Steiner School, UK. Devise a mathematics lesson to have the children calculate the amount of paint required to re-decorate the outside walls of the building. Assume the building is the same on the unseen sides as the seen sides (it isn't). Further assume the building is 20 m wide, 8m deep and 8m up to the first edge of the roof. Assume the apex of the gable end is 12m from the ground. Assume the windows have an area of 2 sq meters. For the purpose of simplifying the exercise, you will need to make some further assumptions. A local builders' merchant sells outside masonry paint that covers $16\ m^2$ for a 5 litre tin. Each 5 ltr tin costs £25. How many tins are needed to paint the whole building and how much will it cost for two coats of paint? Show how you would get the children to devise a mathematical model using algebra for the calculation.

5) Consider Table 1 introduced earlier as well as the attached text on mathematics described by Steiner in "Discussions with Teachers". Using the materials sent, design a 6 week main lesson block on mathematics for a class of your choice. Write a few paragraphs about the age profile (child development stage) and the appropriate teaching & learning method. As well as using this and earlier power points for designing the sequence of learning, you will need to synthesise materials from:

Avison, K & Rawson, M (eds)(2014): *The Educational Tasks and Content of the Steiner Waldorf Curriculum*, Floris Books, chapter 9.

Rawson, M, Burnett, J and Mepham, T (eds) (1999) Steiner Waldorf Education in the UK – Aims, Methods and Curriculum. (Mathematics Components)

Jarman, R (1998): Teaching Mathematics in Rudolf Steiner Schools for Classes I – VIII, Hawthorne Press.

Steiner, (1919): Discussions with Teachers, Anthroposophic Press, chapters 13 & 14.

6) For this exercise you will need a globe, a ruler and string. Using your prior knowledge of the Earth's circumference, take a piece of string and work out the shortest distance of travel between London and Sydney. You will need to make some approximations. What kind of shape does the string take on in your measurements?

7) Your 21 year old sister has gone with friends on a journey to circumnavigate the World. Unfortunately, the last radio message received was when her boat was approximately 250 miles south of the most southern point of New Zealand. Sadly, they appear to have become lost and you are deeply worried. Amongst other things that you do, you decide to send her a prayer. Assuming prayers don't recognise physical boundaries, but they do have to travel, what distance would it have to travel if you live in London? Again, use your globe, prior knowledge and string, with some approximations.